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Simple Coherent State and New Form for Inhomogeneous Differential Realization of OSP(2,1) Superalgebra

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The simple coherent state of the OSP(2,1) superalgebra is constructed and its properties are discussed in detail. The matrix elements of the OSP(2,1) generators in the coherent state space are calculated. The new form for inhomogeneous differential realization of the OSP(2,1) superalgebra is given.

KEY WORDS: Lie algebras; Lie superalgebra; OSP (2,1) superalgebra; cohereent state; representation; inhomogeneous differential realization.

1. INTRODUCTION

Coherent states of Lie (super)algebra have played an important role in the study of quantum mechanics, quantum electrodynamics, quantum optics, and quantum field theory, which provide a natural link between classical and quantum phenomena and are related to the path integral formalism (Fatyga *et al.*, 1991; Quesne, 1990a,b). Quasi-exactly-solvable problems (QESP) in quantum mechanics have become increasingly important because they have been generalized to study the conformal field theory (Shifman and Turbiner, 1989). A connection of QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) was described the first time by Turbiner. Turbiner gave a complete classification of the one-dimensional QESP by making use of the inhomogeneous differential realization of the SU(2) algebra, and pointed out that the multidimensional QESP may be studied and the general procedure to construct the multidimensional QESP in terms of the inhomogeneous differential realizations of the Lie superalgebra was presented (Turbiner, 1988, 1992; Turbiner and Ushveridze, 1987). The key of the solution of the QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Therefore it is very

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important to study the inhomogeneous differential realizations of Lie superalgebras. Some of the inhomogeneous differential realizations are given (Chen, 1993, 2000a–c, 2001). The normalization and unnormalization supercoherent states and the corresponding inhomogeneous differential realizations of the OSP(2,1) superalgebra are constructed (Chen, 2000a–c, 2001). The purpose of the present paper is to derive further the new inhomogeneous differential realization of the OSP(2,1) superalgebra on the basis of studying the simple coherent states. In the present paper we shall first construct the coherent states of the OSP(2,1) superalgebra and discuss their properties. Then we calculate the matrix elements of the OSP(2,1) generators in the coherent state representation and give a new form of the inhomogeneous differential realization of the OSP(2,1) in the coherent-state spaces.

2. THE OSP(2,1) COHERENT STATES AND PROPERTIES

In accordance with Chen (2000a–c) the generators of the OSP(2,1) superalgebra read as

$$\{Q_3, Q_+, Q_- \in \mathrm{OSP}(2,1)_{\bar{0}} \mid \mathrm{V}_+, \mathrm{V}_- \in \mathrm{OSP}(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned} &[Q_3, Q_\pm] = \pm 2Q_\pm, \qquad [Q_+, Q_-] = Q_3, \qquad [Q_3, Q_\pm] = \pm V_\pm \\ &[Q_\pm, V_\mp] = V_\pm, \qquad [Q_\pm, V_\pm] = 0 \\ &\{V_\pm, V_\mp\} = -\frac{1}{4}Q_3, \qquad \{V_\pm, V_\pm\} = \pm\frac{1}{2}Q_\pm \end{aligned} \tag{2}$$

According to Chen (2000a–c) and by relabeling the basis vector $\Phi(k, \alpha)$ of the finite-dimensional irreducible representation of the OSP(2,1) superalgebra by $|N, k, \alpha\rangle$, the actions of the generators on the basis vectors are

$$\begin{aligned} Q_3|N, k, \alpha\rangle &= (-N + 2k + \alpha)\,|N, k, \alpha\rangle \\ Q_+|N, k, \alpha\rangle &= (N - k - \alpha)\,|N, k+1, \alpha\rangle \\ Q_-|N, k, \alpha\rangle &= k|N, k-1, \alpha\rangle \\ V_+|N, k, \alpha\rangle &= \left(-\frac{1}{2}N + \frac{1}{2}k\right)(1-\alpha)\,|N, k, \alpha+1\rangle - \frac{1}{2}\alpha|N, k+1, \alpha-1\rangle \\ V_-|N, k, \alpha\rangle &= \frac{1}{2}k(1-\alpha)\,|N, k-1, \alpha+1\rangle - \frac{1}{2}\alpha\,|N, k, \alpha-1\rangle \end{aligned} \tag{3}$$

where $\{|N, k, \alpha\rangle \mid k + \alpha \leq N,\ N \in Z^+,\ k = 0, 1, 2, \ldots, \alpha = 0, 1\}$ and

$$k = \begin{cases} 0, 1, \ldots, N & \text{when } \alpha = 0 \\ 0, 1, \ldots, N-1 & \text{when } \alpha = 1 \end{cases} \tag{4}$$

The space $\{|N, k, \alpha\rangle\}$ of the irrep N of the OSP(2,1) superalgebra is $2N + 1$ dimensional and may be divided into two subspaces $\{|N, k, 0\rangle\}$ and $\{|N, k, 1\rangle\}$ corresponding to $\alpha = 0, 1$, respectively. All the basis vectors $|N, k, \alpha\rangle$ are assumed to be normalized as

$$\binom{N-\alpha}{k} \langle N, k, \alpha \mid N, k, \alpha\rangle = 1 \tag{5}$$

The completeness condition of the vectors of the irrep may be expressed as

$$\sum_{\alpha,k=0}^{N-\alpha} \binom{N-\alpha}{k} |N, k, \alpha\rangle\langle N, k, \alpha| = I \tag{6}$$

where I is the identity operator.

One can easily show the following formula from Eq. (3):

$$Q_+^n |N, 0, \alpha\rangle = \binom{N-\alpha}{n} n! |N, n, \alpha\rangle \tag{7}$$

where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$.

In terms of Bloch's method we now define the simple coherent state of the OSP(2,1) by applying the exponential operator $\exp(ZQ_+)$ on the lowest-weight state $|N, 0, \alpha\rangle$ of its irrep

$$|Z, \alpha\rangle = C(Z, \alpha) \exp(ZQ_+) \, |N, 0, \alpha\rangle \tag{8}$$

where $C(Z, \alpha)$ is a normalization constant to be determined.

Using the formula (7), the coherent state (8) may be rewritten as follows:

$$|Z, \alpha\rangle = C(Z, \alpha) \sum_{n=0}^{N-\alpha} \binom{N-\alpha}{n} Z^n \, |N, n, \alpha\rangle \tag{9}$$

We require that the OSP(2,1) coherent state defined in this way is normalized in the form

$$\langle Z, \alpha' \mid Z, \alpha\rangle = \delta_{\alpha',\alpha} \tag{10}$$

It follows from Eqs. (9) and (10) that

$$C(Z, \alpha) = (1 + \bar{Z}Z)^{-\frac{1}{2}(N-\alpha)} \tag{11}$$

The scalar product of two coherent states is of the form

$$\langle Z', \alpha \mid Z, \alpha\rangle = C(Z', \alpha') C(Z, \alpha)(1 + \bar{Z}'Z)^{N-\alpha} \delta_{\alpha',\alpha} \tag{12}$$

The expansion coefficients of the coherent state $|Z, \alpha\rangle$ may be found in terms of the complete orthonormal set $\{|N, k, \alpha\rangle\}$. Thus, we have

$$\langle N, k, \alpha \mid Z, \alpha\rangle = C(Z, \alpha) Z^k \tag{13}$$

Although orthogonality is a convenient property for a set of basis vectors, it is not a necessary one. The essential property of such a set is that it be complete. Since the $2N+1$ state vectors $\{|N,k,\alpha\rangle\}$ of an irrep of the OSP(2,1) superalgebra are known to form a complete orthogonal set, the set of the coherent states $\{|Z,\alpha\rangle\}$ for the OSP(2,1) superalgebra can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the complex Z plane, of projection operators of the form $|Z,\alpha\rangle\langle Z,\alpha|$. In order to describe such an integral we introduce generally the differential element of weight area in the Z plane

$$d^2\sigma(Z,\alpha) = \sigma(|Z|,\alpha)\, d^2(Z,\alpha) = \sigma(|Z|,\alpha)\, d(\mathrm{Re}\, Z)\, d(\mathrm{Im}\, Z) \qquad (14)$$

If we set $Z = |Z|\, e^{i^\theta}$, then we may rewrite Eq. (14) as

$$d^2\sigma(Z,\alpha) = \sigma(|Z|,\alpha)\, |Z|\, d\, |Z|\, d\theta \qquad (15)$$

The problem here may by changed to find the weight function $\sigma(Z,\alpha)$ such that

$$\int d^2\sigma\ (Z,\alpha)|Z,\alpha\rangle\langle Z,\alpha| = I \qquad (16)$$

Let $|f\rangle$ and $|g\rangle$ be two arbitrary vectors; then Eq. (16) means that

$$\langle f|g\rangle = \int d^2\sigma\ (Z,\alpha)\langle f|Z,\alpha\rangle\langle Z,\alpha|g\rangle \qquad (17)$$

Substituting Eq. (9) into Eq. (17) and integrating over the entire area of the complex plane, we have

$$\langle f|g\rangle = 2\pi \sum_{\alpha,n=0}^{N-\alpha} \sum_{\alpha,n=0}^{N-\alpha} \binom{N-\alpha}{n}\binom{N-\alpha}{n} \int_0^\infty |Z|^{2n+1}\sigma(|Z|,\alpha)$$
$$\times(1+|Z|^2)^{-N+\alpha}\, d|Z| \times \langle f|N,n,\alpha\rangle\langle N,n,\alpha|g\rangle \qquad (18)$$

Comparing Eq. (18) with Eq. (6) we have

$$2\pi \binom{N-\alpha}{n} \int_0^\infty |Z|^{2n+1}(1+|Z|^2)^{-N+\alpha}\sigma(|Z|,\alpha)\, d|Z| = 1 \qquad (19)$$

With the aid of the following integral identity,

$$\int_0^\infty \frac{x^{2n+1}}{(1+x^2)^m}\, dx = \frac{n!(m-n-2)!}{2(m-1)!} \qquad (20)$$

and by comparing Eq. (19) with Eq. (20) we finally obtain the weight function

$$\sigma(|Z|,\alpha) = \frac{N-\alpha+1}{\pi(1+|Z|^2)^2} \qquad (21)$$

We have thus shown

$$\frac{1}{\pi}\int d^2(Z,\alpha)\,\frac{N-\alpha+1}{(1+|Z|^2)^2}|Z,\alpha\rangle\langle Z,\alpha| = 1 \tag{22}$$

which is a completeness relation for the coherent states of the OSP(2,1) superalgebra of precisely the type desired. As a result of the above completeness relation, an arbitrary vector $|\Psi\rangle$ can be expanded in terms of the coherent states for the OSP(2,1) superalgebra. To secure the expansion of $|\Psi\rangle$ in terms of the coherent states $\{|Z,\alpha\rangle\}$, we multiply $|\Psi\rangle$ by the expression (22) of the unit operator. We then find

$$|\Psi\rangle = \frac{1}{\pi}\int d^2(Z,\alpha)\frac{N-\alpha+1}{(1+|Z|^2)^2}|Z,\alpha\rangle\langle Z,\alpha|\Psi\rangle \tag{23}$$

3. MATRIX ELEMENTS OF THE GENERATORS

The present section will be devoted to calculating the matrix elements of the OSP(2,1) generators in the coherent state representation. The calculation results are as follows:

$$\begin{aligned}
\langle Z',\alpha'|Q_3|Z,\alpha\rangle &= -C(Z',\alpha')\,C(Z,\alpha)(N-\alpha)(1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha}\\
\langle Z',\alpha'|Q_+|Z,\alpha\rangle &= C(Z',\alpha')\,C(Z,\alpha)(N-\alpha)\bar{Z}'(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha}\\
\langle Z',\alpha'|Q_-|Z,\alpha\rangle &= C(Z',\alpha')\,C(Z,\alpha)(N-\alpha)\,Z(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha}\\
\langle Z',\alpha'|V_+|Z,\alpha\rangle &= -\frac{1}{2}(1-\alpha)\mathrm{N}C(Z',\alpha')\,C(Z,\alpha)(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha+1}\\
&\quad -\frac{1}{2}\alpha\,C(Z',\alpha')\,C(Z,\alpha)\bar{Z}'(1+\bar{Z}'Z)^{N-\alpha}\delta_{\alpha',\alpha-1} \qquad (24)\\
\langle Z',\alpha'|V_-|Z,\alpha\rangle &= -\frac{1}{2}\alpha\,C(Z',\alpha')\,C(Z,\alpha)(1+\bar{Z}'Z)^{N-\alpha}\delta_{\alpha',\alpha-1}+\frac{1}{2}(1-\alpha)\\
&\quad \times(N-\alpha)\,C(Z',\alpha')\,C(Z,\alpha)\,Z(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha+1}
\end{aligned}$$

In evaluating the matrix elements, one needs to use only Eqs. (3), (5), and (9); for example,

$$\begin{aligned}
\langle Z',\alpha'|Q_3|Z,\alpha\rangle &= C(Z',\alpha')\,C(Z,\alpha)\sum_{m=0}^{N-\alpha'}\sum_{n=0}^{N-\alpha}\binom{N-\alpha'}{m}\binom{N-\alpha}{n}\\
&\quad\times(\bar{Z}')^m Z^n\langle N,m,\alpha'|Q_3|N,n,\alpha\rangle\\
&= C(Z',\alpha')\,C(Z,\alpha)\sum_{n=0}^{N-\alpha}\binom{N-\alpha}{n}\{-(N-2n-\alpha)\}(\bar{Z}'Z)^n
\end{aligned}$$

$$
\begin{aligned}
&= C(Z', \alpha')C(Z,\alpha)\left\{(N-\alpha)\sum_{n=0}^{N-\alpha}\binom{N-\alpha}{n}(\bar{Z}'Z)^n\right.\\
&\quad\left. -\ 2(N-\alpha)\sum_{n=0}^{N-\alpha-1}\binom{N-\alpha-1}{n}(\bar{Z}'Z)^n\right\}\\
&= -(N-\alpha)C(Z',\alpha')C(Z,\alpha)(1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-\alpha-1}\delta_{\alpha',\alpha}
\end{aligned}
\tag{25}
$$

4. THE INHOMOGENEOUS DIFFERENTIAL REALIZATION OF THE OSP(2,1)

We now consider the actions of the OSP(2,1) generators on the coherent state $\{|Z,\alpha\rangle\}$. By making use of Eq. (3), the completeness relation (6), and the following recurrence formulas

$$
\begin{aligned}
&(N-n)\binom{N}{n} = N\binom{N-1}{n}, \quad (N-n)\binom{N}{n} = (n+1)\binom{N}{n+1},\\
&n\binom{N}{n} = (N-n+1)\binom{N}{n-1}
\end{aligned}
\tag{26}
$$

we easily obtain the following results:

$$
\begin{aligned}
Q_3|Z,\alpha\rangle &= \left\{-(N-\alpha)\frac{1}{1+\bar{Z}Z} + Z\frac{d}{dZ}\right\}|Z,\alpha\rangle\\
Q_+|Z,\alpha\rangle &= \left\{\frac{1}{2}(N-\alpha)\frac{\bar{Z}}{1+\bar{Z}Z} + \frac{d}{dZ}\right\}|Z,\alpha\rangle\\
Q_-|Z,\alpha\rangle &= \left\{\frac{1}{2}(N-\alpha)Z\frac{2+\bar{Z}Z}{1+\bar{Z}Z} - Z^2\frac{d}{dZ}\right\}|Z,\alpha\rangle\\
V_+|Z,\alpha\rangle &= -\frac{1}{2}(1-\alpha)(N-\alpha)\frac{1}{(1+\bar{Z}Z)^{\frac{1}{2}}}|Z,\alpha+1\rangle\\
&\quad -\frac{1}{2}\alpha\frac{1}{(1+\bar{Z}Z)^{\frac{1}{2}}}\left(\frac{1}{2}\bar{Z} + \frac{1}{N-\alpha+1}(1+\bar{Z}Z)\frac{d}{dZ}\right)|Z,\alpha-1\rangle\\
V_-|Z,\alpha\rangle &= \frac{1}{2}(1-\alpha)(N-\alpha)Z\frac{1}{(1+\bar{Z}Z)^{\frac{1}{2}}}|Z,\alpha+1\rangle - \frac{1}{2}\alpha\frac{1}{(1+\bar{Z}Z)^{\frac{1}{2}}}\\
&\quad \times\left\{1+\frac{1}{2}\bar{Z}Z - \frac{1}{N-\alpha+1}(1+\bar{Z}Z)\frac{d}{dZ}\right\}|Z,\alpha-1\rangle
\end{aligned}
\tag{27}
$$

It may be worth noting at this point that many of the foregoing formulas can be abbreviated somewhat by adopting a normalization different from the conventional one for the coherent states. If we introduce the symbol $\| Z, \alpha\rangle$ for the states normalized in the new way and define these as

$$\| Z, \alpha\rangle = \sum_{n=0}^{N-\alpha} \binom{N-\alpha}{n} Z^n |N, n, \alpha\rangle \tag{28}$$

then we may write the scalar product of two such states as $\langle Z', \alpha' \parallel Z, \alpha\rangle$. We see from Eq. (28) that these scalar products are

$$\langle Z', \alpha' \parallel Z, \alpha\rangle = (1 + \bar{Z}' Z)^{N-\alpha} \delta_{\alpha', \alpha} \tag{29}$$

In accordance with the aforementioned consideration, it is clear that we can make $C(z, \alpha) = 1$ in Eq. (9), that is we can obtain a simple inhomogeneous differential realization of the OSP(2,1) superalgebra in the new coherent state space $\{\| Z, \alpha\rangle\}$. We now consider the actions of the OSP(2,1) generators. The results are as follows:

$$\begin{aligned}
Q_3 \| Z, \alpha\rangle &= \left\{ -(N-\alpha) + 2Z \frac{d}{dZ} \right\} \| Z, \alpha\rangle \\
Q_+ \| Z, \alpha\rangle &= \frac{d}{dZ} \| Z, \alpha\rangle \\
Q_- \| Z, \alpha\rangle &= \left[(N-\alpha) Z - Z^2 \frac{d}{dZ} \right] \| Z, \alpha\rangle \\
V_+ \| Z, \alpha\rangle &= -\frac{1}{2}(1-\alpha)(N-\alpha) \| Z, \alpha+1\rangle - \frac{1}{2}\alpha \frac{1}{N-\alpha+1} \frac{d}{dZ} \| Z, \alpha-1\rangle \\
V_- \| Z, \alpha\rangle &= -\frac{1}{2}\alpha \left(1 - \frac{1}{N-\alpha+1} Z \frac{d}{dZ} \right) \| Z, \alpha-1\rangle \\
&\quad + \frac{1}{2}(1-\alpha)(N-\alpha) Z \| Z, \alpha+1\rangle
\end{aligned} \tag{30}$$

It is clear that the aforementioned realizations are inhomogeneous.

5. CONCLUSION

We have constructed the simple coherent state of the OSP(2,1) superalgebra and discussed its properties in detail. We have also calculated the matrix elements of the OSP(2,1) generators. The new inhomogeneous differential realizations of the OSP(2,1) generators have been obtained in the coherent-state space. It may be of use for determining new concrete structure of the quasi-exactly-solvable Hamiltonian corresponding to the OSP(2,1) supersymmetric quantum system.

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